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# MAXIMIZATION OF SYSTEM RELIABILITY WITH LIMITED RESOURCES

by

Lawrence D. Bodin



## OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING



UNIVERSITY OF CALIFORNIA - BERKELEY

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ABSTRACT

The optimization of system reliability of a series parallel system containing $t$ types of components is found where the cost of purchasing the components is disregarded, a component can be assigned to more than one component position, and a limited supply of components is available for assignment. The optimal solution is found by ranking the reliabilities of the components of each type and searching over these ranks in the orders specified in this paper.

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## 1.0 INTRODUCTION

A series parallel arrangement of component positions has $n$ subsystems connected in parallel where the $i^{th}$ subsystem contains $k_i$ component positions joined in series. In this system the $\sum_{i=1}^{n} k_i$ component positions are divided into $t$ types--for example, resistor component positions, transistor component positions, capacitor component positions, etc. A sufficient number of components of each type with not necessarily identical reliabilities are available for assignment so that a feasible assignment of components to component positions can be made. Both the components and the system are subject to one type of failure and the other properties of coherent structures (Barlow and Proschan [1], Birnbaum, Esary, and Saunders [2], Esary and Proschan [3]). The problem is to determine that assignment of components to component positions which maximize the system reliability.

This analysis is applicable in the following type of situation. A manufacturer wishes to produce 100,000 identical series parallel systems each of which is made up of 10 transistors and 15 resistors. From various vendors, the manufacturer can purchase 300,000 transistors with reliability .9, 400,000 transistors with reliability .7, 300,000 transistors with reliability .3, 600,000 transistors with reliability .8, and 900,000 transistors with reliability .4. Thus, each system the manufacturer produces will contain three transistors of reliability .9, four transistors of reliability .7, three transistors of reliability .3, six resistors of reliability .8, and nine resistors of reliability .4. Knowing that a system with a higher reliability commands a higher price, the manufacturer wishes to assign components to component positions in such a way that the reliability of each system is identical and maximal.

## 2.0 Two Path Analysis

A two path series parallel system with path lengths $k_1$ and $k_2$ is given. Let $\ell_{ij}$ denote the number of component positions of type $j$ on path $j = 1, 2, \ldots, t$ . The components of type $j$ available for assignment have reliabilities :

$$p_{1j} \geq p_{2j} \geq \cdots \geq p_{\ell_{1j} + \ell_{2j},\ j},\ j = 1, 2, \ldots, t . \tag{1}$$

For convenience, the number of components of type $j$ available for assignment have been taken equal to the number of component positions of that type. It is shown in Theorem (7) that if there are more components of type $j$ available for assignment than is needed, we need only consider the $\ell_{1j} + \ell_{2j}$ most reliable components.

The following notation is utilized in the ensuing development:

- $A_i, B_i$ - assignments of components to sockets.
- $h_{A_i}\left(p^{A_i}\right) = h_{A_i}$ = reliability of structure under assignment $A_i$ .
- $h_{A_i} \vee h_{A_j} = h_{A_i} + h_{A_j} - h_{A_i} h_{A_j}$
- $A_{ijk}$ - the set of components of type $k$ assigned to path $j$ under assignment $A_i$ .
- $h_{A_{ijk}} = \prod_{c_r \in A_{ijk}} p_{rk}$ . If $A_{ijk} = E$, the empty set, then $h_{A_{ijk}} \equiv 1$ .
- $A_{ij} = \bigcup_{k=1}^{t} A_{ijk}$ - the set of components assigned to path $j$ under assignment $A_i$ .

The analysis of this section is based on the following three assignments:

- $A_1$ - components with reliabilities $p_{1j}, p_{2j}, \ldots, p_{\ell_{1j},\ j}$ , $j = 1, 2, \ldots, t$, assigned to path 1 and the other components to path 2.

- $A_2$ - components with reliabilities $p_{1j}, \ldots, p_{\ell_{2j}, j}$, $j = 1, 2, \ldots, t$, assigned to path 2 and the other components to path 1.
- $A_3$ - an arbitrary assignment.

<u>Lemma 1</u>:

$$h_{A_3} \leq \max\left(h_{A_1}, h_{A_2}\right)$$

<u>Proof</u>:

For $i = 1, 2$, define the following substructure assignments:

$Q_i$ = set of components common to assignments $A_1, A_2, A_3$ on path $i$.

$Q_i = A_{1i} A_{2i} A_{3i}$.

$R_i$ = set of components on path $i$ common to both $A_1$ and $A_3$ but not to $A_2$. $R_i = A_{1i} A_{3i} A_{2,\,3-i}$.

$S_i$ = set of components on path $i$ under $A_3$ and path $3-i$ under $A_1$.

$S_i = A_{3i} A_{1,\,3-i}$.

$T_i$ = set of components on path $i$ common to both $A_2$ and $A_3$ but not to $A_1$. $T_i = A_{2i} A_{3i} A_{1,\,3-i}$.

$U_i$ = set of components on path $i$ under $A_3$ and path $3-i$ under $A_2$.

$U_i = A_{3i} A_{2,\,3-i}$.

The substructures associated with each of these substructure assignments is a series system with independent components; hence, the reliability of each substructure-- $h_{Q_i}, h_{R_i}, h_{S_i}, h_{T_i}, h_{U_i}$ --is the product of the reliabilities of the components in the substructure.

$$h_{A_1} = h_{A_{11}} \vee h_{A_{12}} = h_{Q_1} h_{R_1} h_{S_2} \vee h_{Q_2} h_{R_2} h_{S_1}$$

$$h_{A_2} = h_{A_{21}} \vee h_{A_{22}} = h_{Q_1} h_{T_1} h_{U_2} \vee h_{Q_2} h_{T_2} h_{U_1} \tag{2}$$

$$h_{A_3} = h_{A_{31}} \vee h_{A_{32}} = h_{Q_1} h_{R_1} h_{S_1} \vee h_{Q_2} h_{R_2} h_{S_2} = h_{Q_1} h_{T_1} h_{U_1} \vee h_{Q_2} h_{T_2} h_{U_2} .$$

Then

$$h_{A_3} - h_{A_1} = \left(h_{Q_2} h_{R_2} - h_{Q_1} h_{R_1}\right) \left(h_{S_2} - h_{S_1}\right)$$

$$h_{A_3} - h_{A_2} = \left(h_{Q_1} h_{T_1} - h_{Q_2} h_{T_2}\right) \left(h_{U_1} - h_{U_2}\right). \tag{3}$$

Since the number of components on path 1 (path 2) is fixed and $Q_1$ and $R_1$ ($Q_2$ and $R_2$) are on path 1 (path 2) under assignments $A_1$ and $A_3$, $S_1$ and $S_2$ have the same number of components $m_1$. Hence, $h_{S_2} > h_{S_1}$. By the same argument, the number of components of type $j$, $j = 1, 2, \ldots, t$, on each path is fixed. Hence, the number of components of type $j$, $m_{1j}$, in $S_1$ and $S_2$ is fixed where $\sum_{j=1}^{t} m_{1j} = m_1$. Similarly, $U_1$ and $U_2$ have the same number of components $m_2$ so that $h_{U_1} \geq h_{U_2}$ and the number of components of type $j$, $m_{2j}$, in $U_1$ and $U_2$ is fixed also where $\sum_{j=1}^{t} m_{2j} = m_2$. Define the following sets: $R_{ij} \subset R_i$, $S_{ij} \subset S_i$, $T_{ij} \subset T_i$, $U_{ij} \subset U_i$ where $i$ refers to path number and $j$ refers to component type, $j = 1, 2, \ldots, t$, $i = 1, 2$.

If the lemma is false, then $h_{A_3} > \max\left(h_{A_1}, h_{A_2}\right)$. Thus, from (5.3), it can be concluded that

$$h_{Q_2} h_{R_2} - h_{Q_1} h_{R_1} > 0$$

$$h_{Q_1} h_{T_1} - h_{Q_2} h_{T_2} > 0 , \tag{4}$$

Implying

$$h_{R_2}h_{T_1} = \prod_{j=1}^{t} h_{R_{2j}}h_{T_{1j}} > \prod_{j=1}^{t}\left(h_{R_{1j}}h_{T_{2j}}\right) = h_{R_1}h_{T_2} . \tag{5}$$

If $R_{1j} \subset S_{1j}$ and $T_{1j} \subseteq U_{1j}$ have $\ell_{1j}$ components while $R_{2j} \subset S_{2j}$ and $T_{2j} \subset U_{2j}$ have $\ell_{2j}$ components, then $R_{2j} \subset T_{1j}$ and $R_{1j} \subset T_{2j}$ have $\ell_{1j} + \ell_{2j} - m_{1j} - m_{2j}$ components. Let $v_{1j}$ and $v_{2j}$ be the number of components of type $j$ on path 1 and 2 and assume without loss of generality that $v_{1j} \leq v_{2j}$ . Then, $R_{1j}$ and $T_{2j} \subseteq A_{11}$ and $R_{2j}$ and $T_{1j} \subseteq A_{21}$ where $p_{v_{1j}j} \geq p_{(v_{2j}+1)j}$ . Hence, $h_{R_{2j}} h_{T_{1j}} \leq h_{R_{1j}} h_{T_{2j}}$ for component type $j$ . Thus, inequality (5) is violated and we obtain a contradiction. ||

$A_1$ and $A_2$ are the only assignments that must be considered in determinimg the optimal assignment to a two path series parallel system. Theorem 2 proves that the system reliability under $A_1$ is greater (less) than the system reliability under $A_2$ if the product of the reliabilities of the components in $A_{11}$ is greater (less) than the product of the reliabilities of the components in $A_{22}$ .

**Theorem 2:**

(i) If $h_{A_{11}} > h_{A_{22}}$, then $h_{A_1} > h_{A_2}$ .

(ii) If $h_{A_{11}} < h_{A_{22}}$, then $h_{A_1} < h_{A_2}$ .

(iii) If $h_{A_{11}} = h_{A_{22}}$, then $h_{A_1} = h_{A_2}$ .

**Proof:**

$$h_{A_1} - h_{A_2} = h_{A_{11}} + h_{A_{12}} - h_{A_{21}} - h_{A_{22}} . \tag{6}$$

Since $h_{A_{11}} h_{A_{12}} = h_{A_{21}} h_{A_{22}}$,

$$\frac{h_{A_{11}}}{h_{A_{22}}} = \frac{h_{A_{21}}}{h_{A_{12}}} \tag{7}$$

so that

$$\frac{h_{A_{11}} - h_{A_{22}}}{h_{A_{22}}} = \frac{h_{A_{21}} - h_{A_{12}}}{h_{A_{12}}} \tag{8}$$

Thus,

$$\frac{h_{A_{11}} - h_{A_{22}}}{h_{A_{21}} - h_{A_{12}}} = \frac{h_{A_{22}}}{h_{A_{12}}} > 1 \tag{9}$$

<u>Case I</u>:

If $h_{A_{11}} > h_{A_{22}}$, then $h_{A_{21}} > h_{A_{12}}$ and $h_{A_{11}} - h_{A_{22}} > h_{A_{21}} - h_{A_{12}}$. Hence from (6), $h_{A_1} > h_{A_2}$. This prove (i).

<u>Case II</u>:

If $h_{A_{11}} < h_{A_{22}}$, then $h_{A_{12}}$ and $h_{A_{11}} - h_{A_{22}} < h_{A_{21}} - h_{A_{12}}$. Hence, from (9) and (6) $h_{A_1} < h_{A_2}$. This proves (ii).

<u>Case III</u>:

If $h_{A_{11}} = h_{A_{22}}$, then $h_{A_{21}} = h_{A_{12}}$ so that $h_{A_1} = h_{A_2}$. ||

<u>Corollary 3.</u>

If $\ell_{1j} \leq \ell_{2j}$, $j = 1, 2, \ldots, t$, then $h_{A_1} \geq h_{A_2}$.

Proof:

Under the above hypothesis, $h_{A_{11}} \geq h_{A_{22}}$ so that either (i) or (iii) of Theorem 2 is applicable. ||

Thus, if $\ell_{1j} \leq \ell_{2j}$, $j = 1, 2, \ldots, t$, the most reliable path possible to form is the path containing the smallest number of component positions with the most reliable components of each type assigned to these component positions. Moreover, when there is one type of component (t=1), Corollary 3 is always applicable and the optimal assignment is to assign the most reliable components to the path containing the least number of component positions.

## 3.0 n Path Analysis

Recall that $\ell_{ij}$ denote the number of component positions of type $j$ on path $i$, $j=1,2,\ldots,t$, $i=1,2,\ldots,n$ and the components of type $j$ available for assignment have reliabilities

$$p_{1j} \geq p_{2j} \geq \cdots \geq p_{L_n(j)} \tag{10}$$

where $L_i(j) = \sum_{r=1}^{i} \ell_{rj}$, $i = 0, 1, \ldots, n$, and $L_o(j) = 0$. The following lemma is useful in designing a procedure for optimizing the reliability of the $n$ path system.

Lemma 4:

If $\ell_{ij} \leq \ell_{i+1,\,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, t$, then the optimal assignment $B_1$ is to assign $p_{(L_{i-1}(j)+1),j}$, $p_{(L_{i-1}(j)+2),j}$, $\ldots$, $p_{L_i j}$ on path $i$, $j = 1, 2, \ldots, t$, $i = 1, 2, \ldots, n$.

Proof:

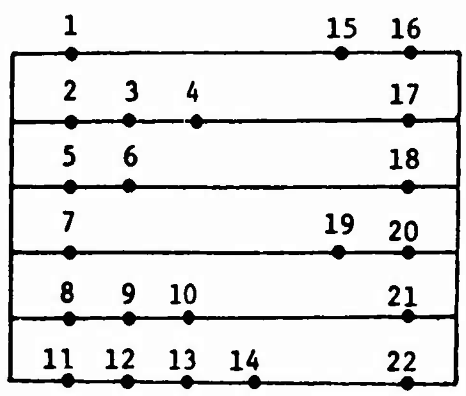
Under $B_1$ the assignment to any two path substructure satisfies Corollary 3 Let $B_2$ be an assignment distinct from $B_1$, that is to say, there exist a two path substructure such that under $B_2$, Corollary 3 is not satisfied. Let $A_1$ be the assignment to this two path substructure before applying Corollary 3 and $A_i$ be the assignment after applying Corollary 3. Then $h_{A_1} < h_{A_2}$. Hence

$$h_{B_2} = 1 - \left(1 - h_{A_3}\right)\left(1 - h_{A_2}\right) < 1 - \left(1 - h_{A_3}\right)\left(1 - h_{A_1}\right) = h_{B_3} \tag{11}$$

where $A_3$ is the assignment under $B_2$ to all paths not in $A_2$ and $B_3$ is the assignment under $A_3$ and $A_1$. Thus, $B_2$ is not optimal and so a contradiction has been found. ||

The procedure given in Lemma 4 initially finds for an n path system, the most reliable path assignment (the path with $\sum_{j=1}^{t} \ell_{1j}$ component positions). It then considers a new system with n-1 paths and repeats the above operation. Thus, at each step in the procedure, the most reliable path assignment with the components yet unassigned is found. This assignment procedure is reversible; i. e. the same assignment can be derived by forming at each step the least reliable path possible to form with the components yet unassignment. Example 1 shows that the procedure alluded to in Lemma 4 and the above discussion does not derive the optimal assignment for all n path series parallel systems with more than one type of component.

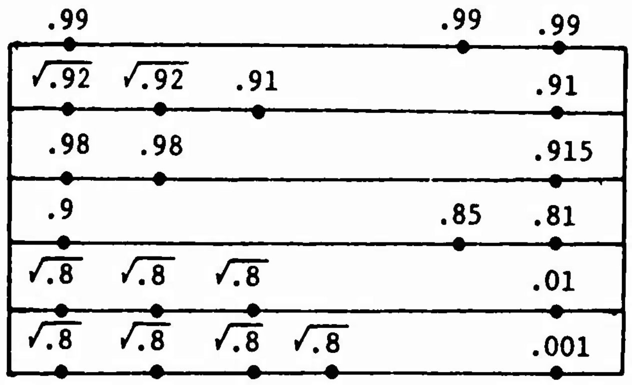
Example 1:



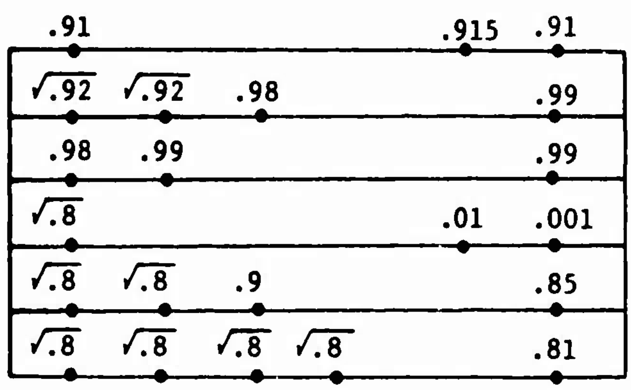

- Component positions 1 - 14 are of Type 1.
- Component positions 15 - 22 are of Type 2.
- Components of Type 1 have reliabilities .99, .98, .98, $\sqrt{.92}$, $\sqrt{.92}$, .91, .90, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$ .
- Components of Type 2 have reliabilities .99, .99, .915, .91, .85, .81, .01, .001 .

Three different assignments $A_1$, $A_2$, $A_3$ are given below. The selection criterion for $A_1$ is to find the most reliable path assignment possible to form with the components yet unassigned and make that assignment. The selection

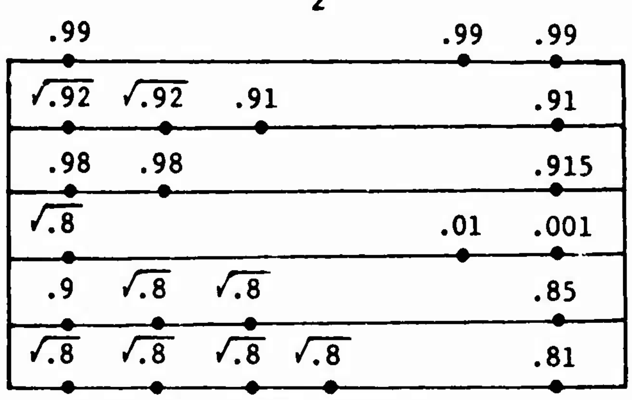
rule for $A_2$ is to find the least reliable path assignment possible to form with the components yet unassigned and make the assignment. The selection rule for $A_3$ is to use the selection rule for $A_1$ on the first three steps and then the selection rule for $A_2$ on the last three steps. The reliabilities are $h_{A_1} = .99973497$, $h_{A_2} = .9998079$, and $h_{A_3} = .9998977$ respectively.




$A_1$



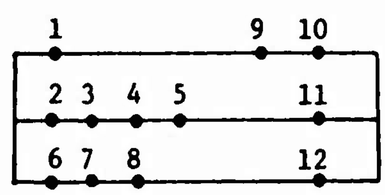

$A_2$




$A_3$

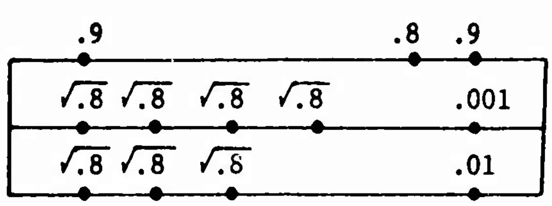
It is shown in Lemma 5 and Theorem 6 that for any n path series parallel system the optimal assignment is contained in the set of assignments $\mathcal{E}_n$ to the n path system generated by either selecting the most reliable or least reliable path assignment in the structure yet to be assigned and making that assignment. For a two path system, this result is proved in Theorem 2 Moreover, for a two path system, $\mathcal{E}_2$ contains but one element, the optimal solution. Thus, for an n path system, only $2^{n-2}$ assignments need be considered if no ties are encountered. If a tie is found, however, both assignments must be considered since a tie cannot be broken arbitrarily (see Example 2).

Example 2:

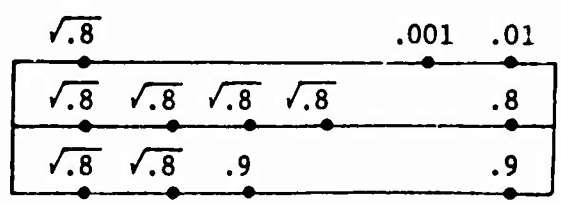



- Component positions 1 - 8 are of Type 1.
- Component positions 9 - 12 are of Type 2.
- Components of Type 1 have reliabilities .9, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$, $\sqrt{.8}$ .
- Components of Type 2 have reliabilities .9, .8, .01, .001 .

By finding the maximum reliability path on the first step a tie occurs between paths 1 and 3. If both assignments are completed, two different assignments $A_1$ and $A_2$ are formed. $h_{A_1} = .660$ while $h_{A_2} = .828$ .




$A_1$




$A_2$

Lemma 5:

If, for an assignment $A_1$ to an $n(\geq 3)$ path series parallel system, there exists a $2 \leq k \leq n-1$ path substructure whose assignment $B_1$ is not optimal to this $k$ path substructure (this means there exists another assignment $B_2$ of the components assigned to this $k$ path substructure which gives a greater reliability), then $A_1$ is not optimal to the $n$ path system.

Proof:

Since $h_{B_1} < h_{B_2}$, $h_{A_1} = 1 - \left(1 - h_{B_3}\right)\left(1 - h_{B_1}\right) < 1 - \left(1 - h_{B_3}\right)\left(1 - h_{B_2}\right)$

where $B_3$ is the assignment under $A_1$ to all paths not in $B_1$. This proves the lemma. ||

Theorem 6:

The optimal assignment $A^*$ to an $n$ path series parallel system is a member of $\mathcal{E}_n$.

Proof:

By induction:

(i) For $n = 2$, it follows from Theorem 2.

(ii) Assume we are given an assignment $A^*$ such that the assignment to all possible substructures with $k$ paths $k = 2,3, \ldots, n-1$, is optimal (Lemma 5) and, hence, contained in its appropriate $\mathcal{E}_k$ (the induction hypothesis). We wish to show that $A^* \in \mathcal{E}_n$.

Since each substructure of $K$ path is a member of its particular $\mathcal{E}_k$, the order of assigning the paths in the substructure is known (although not necessarily unique). The components assigned at each step in the assignment procedure to each substructure dominate

all unassigned components to that substructure in the sense that they have the largest or smallest reliabilities depending upon whether a maximization or minimization operation is carried out. Let

- $D_1$ = assignment of components to the substructure made up of paths 1, 3, 4, ..., n .
- $D_2$ = assignment of components to the substructure made up of paths 2, 3, 4, ..., n .

By the induction hypothesis, since $D_1$ and $D_2$ are in their appropriate $\mathcal{E}_{n-1}$ and the order of assigning components to the paths under $D_1$ and $D_2$ and to any corresponding substructure is known, the order of assigning components to the paths under $D_1$ and $D_2$ are the same until either path 1 is encountered in $D_1$ or path 2 is met is $D_2$ . Let

- $F_1 = \{j \mid \text{path } j \text{ is assigned before path 1 in } D_1 \text{ and before path 2 in } D_2\}$ .
- $F_2 = \{j \mid \text{path } j \text{ is assigned after path 1 in } D_1 \text{ and before path 2 in } D_2\} \cup \{1, 2\}$ .
- $F_3 = \{j \mid \text{path } j \text{ is assigned after path 1 in } D_1 \text{ and after path 2 in } D_2\}$ .

Without loss of generality assume that path 1 is encountered first.

<u>Case 1</u>:

If either $F_1$ or $F_3$ is not empty, the set of components available for assignment is partitioned into two or three parts--those assigned to the paths in $F_1$ , those assigned to the paths in $F_2$ , and those assigned to the paths in $F_3$ . All components assigned to the paths in $F_1$ have reliabilities which are either greater than or less than the components assigned to the paths

in $F_2$ and $F_3$ and all components assigned to the paths in $F_2$ have reliabilities which are either greater than or less than the components assigned to the paths in $F_3$ . By the induction hypothesis, the assignment of the components to the paths in $F_1$ , $F_2$ , and $F_3$ are optimal and the order of assigning the paths to each of the substructures is known. Hence $A^* \in \mathcal{E}_n$ and the order of assignment to $A^*$ is to first assign the components to the paths in $F_1$ following the optimal order, then to the paths in $F_2$ following the optimal order and finally to $F_3$ following the optimal order.

Case 2:

If both $F_1$ and $F_3$ are empty, the $\Gamma_2 = \{1, 2, \ldots, n\}$ . Let path k be the last path assigned to substructure 1, 3, 4, ..., n following the optimal order. Therefore, path k is the next to last path and path 2 is the last path assigned to substructure 2, 3, ..., n . However, path 2 can be assigned before path k in the substructure 2, 3, ..., n and the reliability is unaffected since in a two path substructure it makes no difference which path is assigned first as long as we change the maximization (minimizations) operation to a minimization (maximization)operation on the first step. Hence, if we redefine $F_2 = \{1, 2, \ldots, n\} - \{k\}$ , $F_3 = \{k\}$ , we are in case 1. This completes the induction.

Theorem 7:

If there exist $m_i'$ components of type i available for assignment to an n path series parallel system and $m_i$ sockets $(m_i' \geq m_i)$ , then the optimal assignment uses the $m_i$ most reliable components.

Proof:

For type $i$, let $p_1 \geq p_2 \geq \cdots \geq p_{m_i} \geq p_{m_i+1} \geq \cdots \geq p_{m_i'}$ be the components of type $i$ available for assignment. Let $A_1$ be the optimal assignment in $\mathcal{E}_n$ using components $p_1, p_2, \ldots, p_{m_i}$, $A_2$ be the optimal assignment. in $\mathcal{E}_n$ using an arbitrary subset of the $m_i'$ components and $A_3$ be the assignment which follows the order of $A_2$ but uses components having reliabilities $p_1, p_2, \ldots, p_m$. Then $h_{A_1} \geq h_{A_3}$ and $h_{A_2} - h_{A_3} = \prod_{i=1}^{n} (1 - a_i') - \prod_{i=1}^{n} (1 - a_i)$ where $a_i(a_i')$ is the reliability of the i'th path under assignment $A_2(A_3)$. Since $a_i' \geq a_i$ for each $i$, $h_{A_3} \geq h_{A_2}$. This proves the theorem. ||

The easiest way to compute all members of $\mathcal{E}_n$ is to construct a tree and enumerate all possible cases. At each step in the process, two possible alternatives arise--either to maximize or minimize. An efficient way to carry out the enumeration is to utilize a last in - first out (LIFO) rule. Thus, if the first assignment is to maximize at each step, the next assignment under LIFO is to maximize at each step except the last, the third assignment is to maximize all but the next to the last path and so forth. The tree diagram for a five path system is given in Figure 1. Since the last two steps in the generation of any assignment in $\mathcal{E}_n$ can be found by Theorem 2, each path in the tree requires but three arcs.

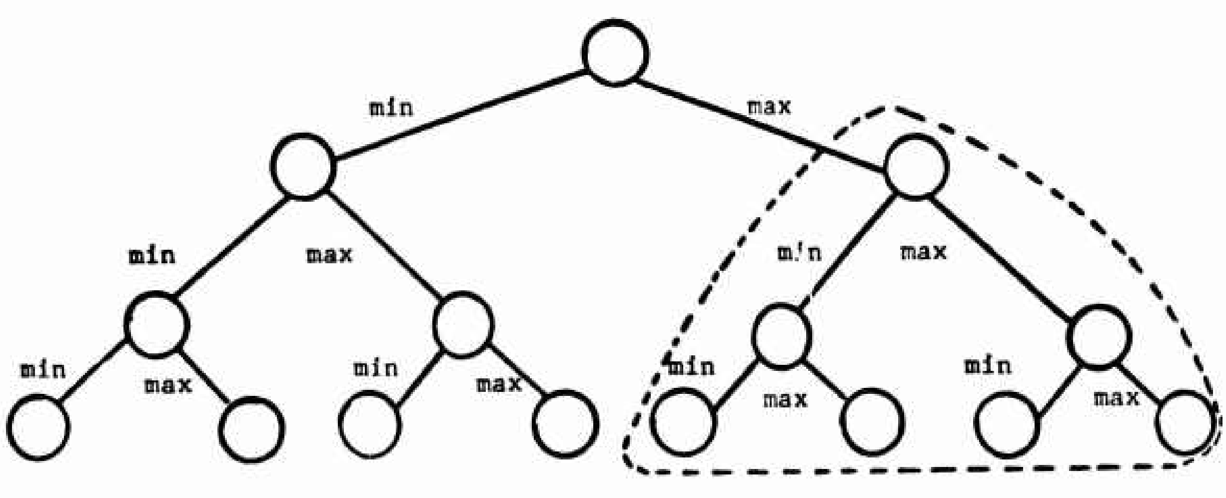



Figure 1

In certain instances, a complete enumeration is not necessary. Suppose that in carrying out the enumeration to the five path system pictured in Figure 1, path 1 is assigned first by maximizing and the sockets on the remaining four paths have the property that $\ell_{ij} \leq \ell_{i+1,j}$, $i = 2,3, \ldots, n-1$, $j = 1,2, \ldots, t$. Then Lemma 4 can be utilized and the entire subtree replaced by a single branch. This subtree is denoted by the dotted circle in Figure 1.

## 4.0 More General Structures

A set of series parallel systems connected in series are given and the sockets in one series parallel system are different from any other series parallel system in the sense that the components to be assigned to one system are independent of those in the other systems. Then the following theorem holds.

Theorem 8:

Under the above assumption, the assignment which maximizes the system reliability can be found by maximizing the reliability of each series parallel subsystem.

Proof:

Let $h_{(i)}$ denote the reliability of the i'th subsystem. Then the reliability of the entire system $h_s$ is

$$h_s = h_{(1)} h_{(2)} \cdots h_{(m)} \tag{12}$$

where $m$ is the number of subsystems. Since each subsystem's assignment is independent of the assignment to any other subsystem

$$\begin{aligned} \text{Max } h_s &= \text{Max } (h_{(1)} \cdots h_{(m)}) \\ &= \text{Max } h_{(1)} \text{ Max } h_{(2)} \cdots \text{Max } h_{(m)} . \end{aligned} \tag{13}$$

This proves the theorem. ||

Thus, $h_s$ is found by $m$ applications of Theorem 6 .

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13. ABSTRACT

The optimization of system reliability of a series parallel system containing t types of components is found where the cost of purchasing the components is disregarded, a component can be assigned to more than one component position, and a limited supply of components is available for assignment. The optimal solution is found by ranking the reliabilities of the components of each type and searching over these ranks in the orders specified in this paper.

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